DEPT. OF MATH./CMA UNIV. OF OSLO
PURE MATHEMATICS NO 3
ISSN 0806–2439 JANUARY 2008

A SEMILINEAR EQUATION FOR THE AMERICAN OPTION IN A GENERAL JUMP MARKET

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ABSTRACT. We study the pricing of American put and call options in a market with jumps. We extend and make rigorous previous work that characterizes the price as a solution of an integro-differential equation set on the whole domain. The equation closely resembles the equation for the corresponding European options, but involves an additional reaction term that depends on the American option value in a nonlinear, non-local and discontinuous manner. Thus standard theory for partial differential equations does not apply, and we give a proper definition of a viscosity solution of the equation. We then show that the characterization is well posed. In particular, we prove a strong comparison principle for the equation using an original approach that overcomes some problems related to the appearance of integrals with respect to unbounded measures. In short, we extend the results in [16] to a general class of exponential additive models. The formulation constitutes a starting point for designing and analyzing "easy to implement" numerical algorithms for computing the value of an American option.

1. INTRODUCTION

The need to incorporate more realistic distributional and path properties in stock price models has enjoyed considerable attention in recent asset pricing literature. Especially, models based on Lévy processes have become popular because of their flexibility and analytical tractability. Lévy processes, i.e. processes with stationary and independent increments, have been studied in the context of financial time series since Mandelbrot [48], and applied to option pricing since Merton [50]. Since then a number of models has spawned. These include the jump-diffusion model by Kou [46], the variance gamma [47], the normal inverse Gaussian [12], tempered stable processes and generalized hyperbolic models [11], [30], [57]. One of the reasons that models based on exponential Lévy processes have become popular is that they have the ability to consistently price traded options across all strikes. However, this can only be achieved for one maturity. To be able to capture all prices across all strikes and maturities consistently, one can loosen the

Date: January 25, 2008.

We thank Fred Espen Benth for suggesting the topic and his efforts in carefully reading and making valuable comments on this paper.

This work was supported in part by the Research Council of Norway through the project "Integro-PDEs: Numerical methods, Analysis, and Applications to Finance" and an Outstanding Young Investigators Award of K. H. Karlsen.

stationarity requirement, as for example in [27], [45]. While these time-inhomogeneous Lévy processes thus better agree with traded option prices, they still have analytical tractability.

In this paper, we consider pricing of American options under such models by considering this as an optimal stopping problem. Classical approaches to solution of optimal stopping problems in finite time horizon can roughly be divided into two categories, namely those based on the *free boundary problem* formulation and those based on *quasi-variational inequality* formulation. In the free boundary problem, one simultaneously looks for the value and a boundary that splits the domain into a continuation set, where the value satisfies a differential equation, and the stopping set, where the value is equal to a known function. The connection between the American option pricing problem and free boundary (or *Stefan*) problems was already given by Samuelson in the ground breaking article [59], with the mathematics worked out by McKean in the appendix [49]. The approach is tricky especially for pure jump processes, since the smooth fit principle typically assumed to hold at the free boundary often brakes down. In this case, the smooth fit should be replaced by a condition of continuous fit. Thus, when stating the problem one always needs to investigate whether continuous or smooth fit should be applied, which can be difficult. The brake down of the smooth fit for Poisson processes was known already by McKean, see also Alili and Kyprianou [1] for a recent survey into these matters. We finally mention that Pham [55] showed that the free boundary formulation can be successfully applied for a strictly positive diffusion coefficient and a finite intensity jump process. The approach through solving quasi-variational inequalities was developed by Bensoussan and Lions [14], [15] and applied to American option pricing by Jaillet et al [53]. In a quite general set up, Pham [56] used this approach to show that the American option price can be characterized as the unique viscosity solution of a fully nonlinear variational inequality.

Even in the case of the classical Black and Scholes market, it seems hard to come to an exact and explicit formula for the American put price, such that numerical values could be computed efficiently. However, it is well known that the price can be expressed quite explicitly in terms of the free boundary. In a recent article, [64] presented an explicit formula as an infinite series in which the terms involve multiple integrals and special functions. This has great value for theoretical and back-testing purposes, but whether the expression gives an efficient tool for computation of numerical values is yet to be tested. Moreover, to our knowledge this formulation or other analytical approximation techniques have not been extended to the general class of exponential additive models (see [13], [22], [26], [64] and the references therein for analytical approximations). One therefore still has to resort to numerical discretization techniques to solve the problem, and since Brennan and Schwartz [20] there has been a lot of work to develop better methods for this purpose. The above frameworks of free boundary and quasi-variational inequalities lend themselves to different numerical schemes, which have advantages and shortcomings specific to the formulation. For these we refer to [51].

Our goal here is to extend a different formulation of the valuation problem carried out in [16], which started from the works of Jamshidian [42] and Kholodnyi [44]. We shall focus on American put options for which the payoff at exercise is given by $g_p(x) = (K - x)^+$, where K is the strike price. Modifications needed to handle the case of a call option $g_c(x) = (x - K)^+$ are also mentioned.

Roughly speaking, in our formulation we seek a function $v = v(t, x)$ solving the following semilinear partial integro-differential equation (PIDE):

$$\mathcal{L}_{BS} v(t, x) + B(t, x, v) = -q(t, x, v), \tag{1.1}$$

where $x \geq 0$, $t \in [0,T)$, $\mathcal{L}_{BS}$ is a differential operator, and B is an integral operator. The nonlinear *reaction term* q takes the form

$$q(t,x,v) = \begin{cases} 0, & g(x) - v(t,x) < 0, \\ c(t,x,v), & g(x) - v(t,x) \geq 0. \end{cases}$$

for a *cash flow function* $c = c(t,x,v)$ defined as

$$c(t,x,v) = \Big(rK - dx - D_g(t,x,v)\Big)^+,$$

for the put option and

$$c(t,x,v) = \Big(dx - rK - D_g(t,x,v)\Big)^+,$$

for the call option, where D_g is another integral operator depending on the payoff g and $r \geq 0$, $d \geq 0$ are the constant interest and dividend rates, respectively. In addition, the value satisfies the terminal condition $v(T,x) = g(x)$. The exact form of the operators can be found in sections 5 and 6. We call this the *semilinear Black and Scholes* (SLBS) equation. In the rest of the article, we shall drop the dependence of the integral operator $D = D_g$ on the payoff g, as we will mainly deal with the case of a put option.

One of our main motivations for studying the SLBS equation is that it allows to design and analyze "easy to implement" numerical schemes. Notice that we could regain the PIDE for the price of a European option by simply taking away the reaction term q. In fact, any solver for the European price can be turned into a solver for the American price using the semilinear formulation. Thus, the equation is also referred to as the nonhomogeneous Black and Scholes equation in the literature. Simple examples of such schemes for the Black and Scholes market were studied in Benth et al [17], where convergence proofs were given along the lines of Barles and Souganidis [7], [10]. Our formulation is also related to so called penalty schemes, which have been studied in connection to American option pricing in [33], [65], [52], as some of these schemes can be seen as approximations to the semilinear equation (1.1).

Notice that our equation is set in the whole domain $[0,T) \times \mathbb{R}_+$, so we do not need to determine a free boundary. In addition, there are no side constraints as in the quasi-variational formulation. However, the nonlinearity $v \mapsto q(t,x,v)$ is *discontinuous*, which raises the question how one should interpret the semilinear equation. Guided by the dynamic programming principle, we suggest a suitable definition of a *viscosity solution* (see [29], [63]) for the semilinear PIDE (1.1). Even if the application of viscosity solutions theory for control problems is standard by now, dealing with a discontinuous operator is not. Here we apply ideas from Benth et al [16], which again draws from the work of Ishii [36] for first order differential operators.

One of the main contributions here is our proof of the comparison principle for the SLBS equation. The Lévy measure of the integral operators in the equation may have a second order singularity at zero, so it is not always obvious whether such integrals are well defined. This makes the application of maximum principle for semicontinuous functions (also known as Ishii's lemma, see [28] and [29]) in connection with integro-differential equations problematic, and there has been increasing interest in this issue. We refer the interested reader to [40] for details. To gain more insight, we go back to the original approach of using semiconvex approximations. This dates back to the early work of Jensen, Ishii and Lions in [41], [37] and [38], see also the textbook of Yong and Zhou [63]. For an adaption to the nonlocal setting, our main source of inspiration is Jakobsen and Karlsen [40]. The approach has some advantages to it, some of which seem original. First, there is no need for an abstract maximum principle as the proof uses the second order conditions for maxima from standard multivariable calculus. Moreover, there is no need to decompose the integral operators into parts separating the singular region from the rest, which to our knowledge is a new feature compared to all the previous work done on Hamilton-Jacobi equations involving singular integral terms.

A similar equation has been used in several articles to study numerical methods under various model assumptions. For example, essentially the same PIDE in the case of variance gamma has already appeared in Hirsa and Madan [35], see also Carr and Hirsa [21] where a transformed equation is used in connection to model calibration. In addition, in a series of papers [23], [24], [25], Chiarella and Ziogas combine such an equation with the incomplete Fourier transform to derive new numerical schemes. In all of these articles, the equation is stated in the form of a free boundary problem. However, no rigorous theory is built. We argue that such a formulation can not be stated on the whole domain because of lack of smoothness of the solution over the free boundary. Moreover, in our formulation no precomputation of a free boundary is needed in order to solve the option price.

Let us note that in a complete market setting (the pure diffusion case), the reaction term q is nothing but the consumption density process of the writer of the option. Thus the equation should be interpreted as the infinitesimal version of the early exercise premium representation of the American option price. See the last section in [16] for a heuristic discussion of this point. Finally, while we do not study the perpetual case $T = +\infty$ here, one can see that the price of a perpetual option should satisfy an elliptic version of the semilinear equation.

The remainder of this article is organized as follows. In section 2 we establish some notations, and section 3 offers a brief introduction to exponential additive models. In section 4 we review results on optimal stopping, and we show on a heuristic level how to derive a semilinear equation for the American put option price in section 5. Then we set up a rigorous definition of a solution to this equation in section 6 via viscosity solution theory. Finally, sections 7 and 8 give the main results on well-posedness of the American option value in our framework.

2. Some notations

Let us fix some notations on classes of functions we will be working with. For a set $A \subset \mathbb{R}^N$, $N \in \mathbb{N}$, let $B(A)$ be any class of real valued functions on A. We will denote by $B_1(A)$ the subclass of functions with at most linear growth at infinity, that is functions $f \in B(A)$ such that

$$f(x) \leq L(1 + |x|) \tag{2.1}$$

for some $L > 0$. We recall that for every locally bounded function $f : A \mapsto \mathbb{R}$, its *upper* and *lower semicontinuous envelopes*, denoted by f^* and f_* respectively, are defined as

$$h^*(x) := \limsup_{y \to x} h(y), \quad h_*(x) := \liminf_{y \to x} h(y).$$

A locally bounded function $f : A \mapsto \mathbb{R}$ is said to be *upper semicontinuous* if $f^* \leq f$ and *lower semicontinuous* if $f_* \geq f$. Especially,

$$H^*(x) := H(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$$

$$H_*(x) := \begin{cases} 0 & \text{if } x \leq 0 \\ 1 & \text{if } x > 0 \end{cases}$$

are the upper and lower semicontinuous envelopes, respectively, of the Heaviside function H. If h is both upper and lower semicontinuous then it is continuous. We denote the sets of upper and lower semicontinuous functions by $USC(A)$ and $LSC(A)$, respectively. As usual, we denote by $C(A)$ the class of continuous functions on A. In addition we denote by

$USC_1^+(A)$ ($LSC_1^+(A)$) the class of non-negative functions belonging to $USC(A)$ ($LSC(A)$) and satisfying (2.1).

Let $\mathbb{R}_+ := [0,\infty)$. In the following sections we let $\mathcal{O}_T = [0,T) \times [0,\infty)$, and $\overline{\mathcal{O}_T} = [0,T] \times [0,\infty)$ then denotes the time-space domain on which functions are defined. We say that a function v is $C^{1,2}$ at the point $(t,x) \in \mathcal{O}_T$ if there are $(p,P) \in \mathbb{R}^n \times \mathcal{S}^n$ such that

$$v(x+y) = v(x) + \langle p, y\rangle + \frac{1}{2}\langle Py, y\rangle + o(|y|^2),$$

and $C_1^{1,2}$ at (t,x) if, in addition, it has at most linear growth so that (2.1) is satisfied. Finally, $C_1^{1,2}(\mathcal{O}_T)$ is the class of functions that are $C_1^{1,2}$ at all $(t,x) \in \mathcal{O}_T$.

3. Exponential additive processes

In this section we briefly review the class of exponential additive processes we will use to model stock price evolution. We rely largely on [45], which also introduces some financial applications. General references for the special case of Lévy processes are [2],[18], [60], and financial applications are discussed, for example, in [27], [31] and [57]. Properties of additive processes can be found in chapter 2 of [60] and chapter 14 of [27]. Relations to semimartingales are detailed in [39]. Let us first, however, note that financial models driven by such processes are in general incomplete, meaning that not all derivatives can be perfectly replicated by dynamic trading in the underlying. This then implies that there are in fact an infinite number of equivalent martingale measures to choose from, each giving an arbitrage free pricing rule. While there are several theoretical and practical ways to choose one, we simply assume in this paper that a pricing measure $\mathbb{Q}$ is given and all the dynamics considered henceforth are under this measure.

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{Q})$ be a filtered probability space satisfying the usual conditions. A stochastic process $X = (X(t))_{t\in[0,T]}$ on $\mathbb{R}$ is called *additive* if it is stochastically continuous with RCLL (i.e. right continuous with left limits) sample paths and independent increments. Given such a process X we assume that $\mathbb{F} = \mathbb{F}^X$, i.e. we take the filtration to be the completed natural filtration generated by X.

Some additive processes are not semimartingales: any deterministic, continuous function with infinite variation provides a trivial example of this. This is not desirable since we lose Itô's formula and further, we might introduce models with arbitrage opportunities. It is furthermore clear that excluding such peculiarities from our modeling framework is not restricting us in building realistic models. This motivates us to work with a slightly more restricted class of processes.

Definition 3.1. The process X has *independent increments with absolutely continuous characteristics*: that is, for every $t \in [0,T)$ the distribution of $X(s) - X(t)$, $t < s$ is independent of $\mathcal{F}_t$ and the characteristic function $\Phi_t(u) := \mathbb{E}[\exp(iuX_t)]$ of $X(t)$ is given by

$$\Phi_t(u) = \exp\Big\{ \int_0^t \Big(iub(s) - \frac{1}{2}u^2\sigma^2(s) + \int_{\mathbb{R}} (e^{iuz} - 1 - iuz)\nu_s(dz)\Big) ds \Big\}. \tag{3.2}$$

Here b, σ are measurable functions on $[0,T]$ and for each s, $\nu_s(\cdot)$ is a Borel measure on $\mathbb{R}$ such that $\nu_s(0) = 0$,

$$\int_0^T \Big(|b(s)| + |\sigma^2(s)| + \int_{\mathbb{R}} (z^2 1_{|z|\le 1}(z) + \exp(2z) 1_{\{|z|>1\}}(z))\nu_s(dz)\Big) ds < \infty.$$

A stochastic process with independent increments and absolutely continuous characteristics is henceforth abbreviated PIIAC.

Stochastic continuity of X is actually implied by equation (3.2). Furthermore, a PIIAC is an additive process in law and has an RCLL modification which is also a semimartingale, see [45]. We will always work with this RCLL version of X. Finally, it follows from Corollary 4.18 in [39] that X is also quasi-left-continuous, i.e. left continuous over stopping times.

In the definition above, the integrability condition on the tails of the measure $\nu_s(\cdot)$ is stronger than what is usually given. This assumption is related to our proof of the comparison principle for solutions of (1.1), and also implies that the price process is square-integrable. Notice, however, that we allow for fully general behavior of the measure near zero, and a possibly vanishing σ to include pure jump processes with infinite activity. In addition, to make it easier to take limits we will require that b and σ are continuous functions on $[0,T]$, and $\nu_s(\cdot) = \rho(s)\nu(\cdot)$ for a continuous function ρ and a time independent measure $\nu(\cdot)$. Then we also have that if $f = f(z)$ is a continuous function and $\kappa \geq 0$ is such that

$$\int_{|z|>\kappa} f(z)\nu_s(dz) < \infty \qquad \text{for all } s \in [0,T],$$

then

$$\lim_{s\to t} \int_{|z|>\kappa} f(z)\nu_s(dz) = \int_{|z|>\kappa} f(z)\nu_t(dz).$$

Let $J_X(ds,dz)$ denote the (random) *jump measure* associated to the RCLL process X (see [39]) and let

$$\tilde{J}_X(ds,dz) = J_X(ds,dz) - \nu_s(dz)ds$$

denote the compensated jump measure. Given our assumptions on X, it is a *special* semimartingale and thus has *canonical representation* (see [39], II.2.34 or [45])

$$X(t) = \int_0^t b(s)ds + \int_0^t \sigma(s)dW(s) + \int_0^t \int_{\mathbb{R}\setminus\{0\}} z\tilde{J}_X(ds,dz), \tag{3.3}$$

where W is a Brownian motion. Then we model the dynamics of the stock price $(S(t))_{t\in[0,T]}$ under the martingale measure $\mathbb{Q}$ as

$$S(t) = S(0)\exp((r-d)t + X(t)). \tag{3.4}$$

Assuming that $\mathbb{Q}$ is a martingale measure means that the discounted price process with dividends

$$\bar{S}(t) = e^{-(r-d)t}S(t) = S_0\exp(X(t))$$

has to be a (local) martingale under $\mathbb{Q}$. Then the price model is free of arbitrage. A necessary and sufficient condition for the martingale property to hold is that, for each t the characteristics $(b(t), \sigma^2(t), \nu_t(\cdot))$ satisfy

$$b(t) + \frac{\sigma^2(t)}{2} + \int_0^t \int_{\mathbb{R}} (e^z - 1 - z)\nu_s(dz)ds = 0. \tag{3.5}$$

For example, in the Black-Scholes model $d = 0$, $\nu \equiv 0$, $\sigma(t) \equiv \sigma$ so we must have $b(t) \equiv -\frac{1}{2}\sigma^2$, which combined with (3.4) gives the risk neutral drift $r - \frac{1}{2}\sigma^2$ for the log-prices of this fundamental model. Finally, we will assume without loss of generality that the the model satisfies the natural condition

$$\mathbb{Q}^{t,x}(\tau_A \leq T) > 0, \tag{3.6}$$

for any $t < T$, $x > 0$, and any open set A such that $cl(A) \subset (0,+\infty)$ where $\tau_A := \inf\{u \in [t,T] : S(u) \notin A\}$ is the first exit time from the set.

In the next section, we will use that by independence of increments of X the price process S is a strong Markov Process. This is usually proved for Lévy processes only, but it holds for the class PIIAC also as is argued on page 267 in [34]. Since X is a real valued, quasi-left-continuous strong Markov Process with RCLL paths on $[0,T]$, it is a *standard* Markov process in the sense of Blumenthal and Getoor [19].

4. Optimal stopping of Markov Processes

In this section we state some general results in optimal stopping of strong Markov Processes. We follow Shiryaev [61] which includes the proofs in the time homogeneous case - for the nonhomogeneous case the claims can be seen to hold by considering the corresponding space-time process $(t, S(t))$. We also recommend the recent book by Peskir and Shiryaev [54]. Let S be a standard Markov process with associated transition function $(s,y) \to \mathbb{Q}^{s,y}$ (see [58] or [61] for rigorous definitions). Then, for each fixed (t,x), $\mathbb{Q}^{t,x}$ is a probability measure such that $\mathbb{Q}^{t,x}(S(t)=x)=1$, and we denote by $\mathbb{E}^{t,x}$ the expectation under this measure.

Given $g \in C_1(\mathbb{R}_+)$, $g \geq 0$, we wish to find

$$v(t,x) = \sup_{t \leq \tau \leq T} \mathbb{E}^{t,x}[e^{-r(\tau-t)} g(S(\tau))], \tag{4.1}$$

where, especially, $g(x) = (K-x)^+$ for the put option and $g(x) = (x-K)^+$ for the call option. In the financial context, any stopping time is an exercise strategy of the American option. It is then of natural interest also to look for a stopping time τ_0 which achieves the maximal expectation, that is

$$v(t,x) = \mathbb{E}^{t,x}[e^{-r(\tau_0-t)} g(S(\tau_0))].$$

If such a τ_0 exists, it is called an optimal stopping time.

From the definition it follows immediately that $v \geq g$, $v(T,x) = g(x)$, and the optional stopping theorem together with the martingale property of $\bar{S}$ implies that the value function v satisfies

$$v(t,x) \leq L(1+x) \tag{4.2}$$

for general $g \in C_1(\mathbb{R}_+)$, or $v(t,x) \leq K$ for the put and $v(t,x) \leq x$ for the call option especially. Also, if $g \geq 0$ is not identically zero, then it follows from (3.6) that $v > 0$ on $(0, +\infty)$. To apply general theorems in optimal stopping, it is required that the process $g(S_t)$ satisfies some stronger integrability conditions. For example, in [61] it is assumed that

$$\mathbb{E}[\sup_{t \in [0,T]} |g(S_t)|] < \infty. \tag{4.3}$$

In the case of a put option this is trivially satisfied, but the case of a call option depends, at least a priori, on integrability of S. For a square integrable S, such as our price process defined in the previous section, the condition is seen to hold by applying Doob's martingale inequality to $\bar{S}$.

The next proposition is used heavily both in the next section when deriving the semilinear equation (1.1) and in section 7 where it is used to show that the value function v is a viscosity solution of (1.1). For $\epsilon \geq 0$, define the stopping time

$$\tau_\epsilon := \inf\{u \in [t,T] | v(u, S(u)) \leq g(S(u)) + \epsilon\}.$$

Proposition 4.4 (Dynamic programming principle (DPP) for optimal stopping)**.**

(i) *For all stopping times θ taking values in $[t,T]$, we have*

$$v(t,x) \geq \mathbb{E}^{t,x}[e^{-r(\theta-t)} v(\theta, S(\theta))]. \tag{4.5}$$

(ii) *Any stopping time $t \leq \theta \leq \tau_\epsilon$ satisfies*

$$v(t,x) = \mathbb{E}^{t,x}[e^{-r(\theta-t)}v(\theta, S(\theta))]. \tag{4.6}$$

(iii) *τ_0 is an optimal stopping time for $g(S(t))$, and $e^{-r(u\wedge\tau_0-t)}v(u\wedge\tau_0, S(u\wedge\tau_0))$ is a martingale.*

Proof. The inequality (4.5) follows from the r-excessivity of the value function v, see Theorem 1 and Lemma 1 in chapter three of [61]. The opposite inequality follows by the ϵ-optimality of τ_ϵ ([61], p.126, Lemma 10). That τ_0 is an optimal stopping time follows from [61], p.137, Theorem 6. This fact then implies the martingale property, see appendix D in [43] and the references therein. □

Since τ_0 is optimal and $v \geq g$ everywhere, we will call the set $\mathcal{C} := \{(t,x) \in \overline{\mathcal{O}_T} : v(t,x) > g(x)\}$ the *continuation region* and $\mathcal{S} := \{(t,x) \in \overline{\mathcal{O}_T} : v(t,x) = g(x)\}$ the *stopping region*.

In addition to satisfying the growth properties already stated, we only need the solutions to be continuous to apply viscosity solution theory.

Proposition 4.7. *Suppose the Markov process is an exponential additive process as given in section 3. Suppose furthermore the pay-off function $g \in C_1(\mathbb{R}_+)$, $g \geq 0$ is Lipschitz. Then the value function v in (4.1) is continuous.*

Proof. This is proved by Pham [56] for the case with time-independent measure ν. For the time dependent case $\nu_t(\cdot) = \rho(t)\nu(\cdot)$ the same proof holds using continuity (and thus boundedness) of the function $\rho(\cdot)$. □

Finally, we note that the value function is classically characterized as the smallest superharmonic majorant of the pay-off g. We will see in section 8 that the class of supersolutions of equation (1.1) satisfies an analogous result.

5. Formal derivation of the semilinear equation

Next we will proceed to derive the semilinear Black-Scholes equation for the American put option $g(x) = (K-x)^+$. Our derivation here is only formal, rigorous definitions and proofs follow in the subsequent chapters. We assume especially that $v \in C^{1,2}(\mathcal{O}_T)$.

Let S be an exponential PIIAC process under the risk neutral measure as defined in section 3, and let $(t,x) \in \mathcal{O}_T$. Applying Itô's formula to the process

$$Y(s) := e^{-r(u-t)}v(u, S^{t,x}(u)), \quad u \in [t,T]$$

yields

$$\begin{aligned} dY(s) &= e^{-r(u-t)}[\mathcal{L}_{BS}v(u,S^{t,x}(u)) + B(u,S^{t,x}(u),v)]du \\ &+ e^{-r(u-t)}\sigma(u)S(u)\partial_x v(u,S^{t,x}(u))dW_u \\ &+ e^{-r(u-t)}\int_{\mathbb{R}} v(u,S(u-)e^z) - v(u,S(u-))\tilde{J}_X(dt,dz), \end{aligned}$$

where $\mathcal{L}_{BS}v(u,s) = \partial_u v(u,s) + (r-d)s\partial_s v(u,s) + \frac{1}{2}\sigma^2(t)s^2\partial_s^2 v(u,s) - rv(u,s)$ and

$$B(u,s,v) = \int_{\mathbb{R}} [v(u,se^z) - v(u,s) - s(e^z-1)\partial_s v(u,s)]\nu_u(dz).$$

This integral is well defined for $v \in C^{1,2}(\mathcal{O}_T)$, as can be seen by Taylor's theorem and the fact that the measure $\nu_t(\cdot)$ integrates $(e^z-1)^2$ on $\mathbb{R}\setminus\{0\}$. The stochastic integrals are true martingales with zero expectation, at least up to an exit time from a small neighborhood of (t,x). Taking expectations on both sides and using inequality (4.5) then gives

$$\mathcal{L}_{BS}v(t,x) + B(t,x,v) \leq 0 \tag{5.1}$$

everywhere for the value function. Furthermore, equation (4.6) implies

$$\mathcal{L}_{BS}v(t,x) + B(t,x,v) = 0. \tag{5.2}$$

in the continuation region $\{v(t,x) > g(x)\}$. In the exercise region $\mathcal{L}_{BS}v(t,x)+B(t,x,v)$ is non-positive. However, as we will see, it is possible to derive a lower bound for $\mathcal{L}_{BS}v(t,x)+B(t,x,v)$ in this region as well. To this end, fix a point (t,x) in the exercise region. Since $v(t,x) = g(x)$ and $v \geq g$ everywhere, (t,x) is a global maximizer of $g - v$. In what follows, we consider the put option. Because $v > 0$ and $g(x) = 0$ for $x \geq K$, we conclude that $x < K$, where g is smooth. We must have

$$\partial_t v(t,x) = 0, \quad \partial_x v(t,x) = -1, \quad \partial_x^2 v(t,x) \geq 0.$$

Recalling that

$$H_*(x) := \begin{cases} 0 & \text{if } x \leq 0 \\ 1 & \text{if } x > 0 \end{cases}$$

the integral term has the value

$$\begin{aligned} B(t,x,v) &= \int_{\mathbb{R}_0} v(t,xe^z) - (K - xe^z)\nu_t(dz) \\ &= \int_{\mathbb{R}_0} H_*(v(t,xe^z) - (K-xe^z))(v(t,xe^z) - (K-xe^z))\nu_t(dz), \end{aligned}$$

where the last equality follows from noticing that $v(t,xez) \geq g(xe^z) \geq (K - xe^z)$. Thus we have discovered that

$$\begin{aligned} &\mathcal{L}_{BS}v(t,x) + B(t,x,v) \\ &\geq -\Big(rK - dx - \int_{\mathbb{R}} H_*(v(t,xe^z) - (K-xe^z))(v(t,xe^z) - (K-xe^z))\nu_t(dz)\Big). \end{aligned}$$

However, since (5.1) tells us that the right hand side in the above inequality is nonpositive, we conclude that

$$\begin{aligned} &\mathcal{L}_{BS}v(t,x) + B(t,x,v) \\ &\geq -\Big(rK - dx - \int_{\mathbb{R}} H_*(v(t,xe^z) - (K-xe^z))(v(t,xe^z) - (K-xe^z))\nu_t(dz)\Big)^+ \end{aligned} \tag{5.3}$$

when $v(t,x) = g(x)$.

Let us now collect the information revealed by the derivations and the remark above into a single equation, without explicitly using the concept of a free boundary. For v, let

$$D(t,x,v) := \int_{\mathbb{R}} H_*(v(t,xe^z) - (K-xe^z))(v(t,xe^z) - (K-xe^z))\nu_t(dz). \tag{5.4}$$

It is not obvious when D has a finite value for a given function v and point (t,x). We treat this question in detail in the next section. Now, we define the *cash flow function*

$$c(t,x,v) = (rK - dx - D(t,x,v))^+ \tag{5.5}$$

and the *reaction term*

$$q(t,x,v) = H(g(x) - v(t,x))c(t,x,v). \tag{5.6}$$

Then the semilinear Black and Scholes partial integro-differential equation for the value function of an American option is

$$\mathcal{L}_{BS}v(t,x) + B(t,x,v) = -q(t,x,v). \tag{5.7}$$

As noted in section 4, the value function also satisfies the terminal condition

$$v(T,x) = g(x). \tag{5.8}$$

We should explain in what sense exactly can (5.7) be taken as an equality. On the one hand, if (t,x) belongs to the *interior* of the stopping region $\mathcal{S}$, then $v(s,y) = K - y$ in a neighborhood of (t,x) and the inequality in (5.3) becomes an equality. On the other hand, the continuation region is known to be open, non-empty and connected. Thus (5.7) holds almost everywhere on $\mathcal{O}_T$. However, this characterization is not unique without further knowledge of the behavior of v at the boundary of $\mathcal{C}$. In the viscosity solutions approach presented in the next section the inequalities derived above are built into the definition of a solution and no such information is needed.

One derives the equation for the call option similarly as above. We point out that for the call option, it is well known that if there are no dividends ($d = 0$) we have

$$v(t,x) = \mathbb{E}^{t,x}[e^{-r(T-t)}g(S(T))]$$

That is, the value of the American call option under $\mathbb{Q}$ equals the value of the European option under $\mathbb{Q}$, and it is not optimal to exercise before the terminal time T.

6. Viscosity solutions

In the previous section, we derived a partial integro-differential equation for the value of an American put. However, it is known that the value function is not in general smooth. Also, the discontinuity of the nonlinear operator in the solution v is nonstandard, and we need to interpret equation (5.7) in a proper way. To deal with these problems, we follow [16] and use the framework of viscosity solutions theory. In addition, care has to be taken to insure the integral term D appearing in the cash flow function c is well defined.

For a function v which is $C_1^{1,2}$ at (t,x), define

$$\overline{D}(t,x,v) := \int_{\mathbb{R}} H^*(g(x) - v(t,x))H_*(v(t,xe^z) - g(xe^z))(v(t,xe^y) - (K - xe^z))\nu_t(dz)$$

and

$$\underline{D}(t,x,v) := \int_{\mathbb{R}} H_*(g(x) - v(t,x))H_*(v(t,xe^z) - g(xe^z))(v(t,xe^y) - (K - xe^z))\nu_t(dz).$$

We discuss the finiteness of these integrals after giving our definition of viscosity solutions. Given the above definitions, we denote the corresponding source terms by

$$\begin{aligned} q^*(t,x,v) &:= H^*(g(x) - v(t,x))\Big(rK - dx - \overline{D}(t,x,v)\Big)^+ \\ &= H^*(g(x) - v(t,x))\Big(rK - dx - D(t,x,v)\Big)^+, \end{aligned}$$

and

$$\begin{aligned} q_*(t,x,v) &:= H_*(g(x) - v(t,x))\Big(rK - dx - \underline{D}(t,x,v)\Big)^+ \\ &= H_*(g(x) - v(t,x))\Big(rK - dx - D(t,x,v)\Big)^+. \end{aligned}$$

We warn the reader that these are definitions, and despite the notation we do not yet claim any semicontinuity properties of q^*, q_* but instead return to these questions later in this section. Let us note that the equalities with D replacing $\underline{D}, \overline{D}$ make sense. Since $g(xe^z) \geq K - xe^z$, the integrand of D is nonnegative everywhere, and thus the integral is well defined in the Lebesgue sense even though it could take infinite values. We can extend the domain of definition for $(\cdot)^+$ to the extended real line $[-\infty, +\infty]$ by setting $(-\infty)^+ = 0$. It will be shown shortly that D is finite in the region where $g(x) \geq v(t,x)$. In the region where $g(x) < v(t,x)$ (and D could be infinite so that c vanishes), H^* and H_* vanish so the equalities still hold. However, in the development of the theory, it will be convenient to work with $\underline{D}$ and $\overline{D}$.

Definition 6.1. (i) A non-negative function $v \in USC_1^+(\overline{\mathcal{O}_T})$ is a *viscosity subsolution* of (5.7) if and only if for all $\phi \in C_1^{1,2}(\overline{\mathcal{O}_T})$ such that $v \le \phi$ we have:

$$\mathcal{L}_{BS}\phi(t,x) + B(t,x,\phi) + q^*(t,x,\phi) \ge 0 \tag{6.2}$$

whenever $\phi(t,x) = v(t,x)$ and $v(t,x) > 0$. If, in addition, $v|_{\{t=T\}} \le g$ on $[0,\infty)$, then v is a viscosity subsolution of the terminal problem (5.7)-(5.8).

(ii) A non-negative function $v \in LSC_1^+(\overline{\mathcal{O}_T})$ is a *viscosity supersolution* of (5.7) if and only if for all $\phi \in C_1^{1,2}(\overline{\mathcal{O}_T})$ such that $v \ge \phi$ we have:

$$\mathcal{L}_{BS}\phi(t,x) + B(t,x,\phi) + q_*(t,x,\phi) \le 0. \tag{6.3}$$

whenever $v(t,x) = \phi(t,x)$. If, in addition, $v|_{\{t=T\}} \ge g$ on $[0,\infty)$, then v is a viscosity supersolution of the terminal problem (5.7)-(5.8).

(iii) A non-negative function $v \in C_1(\overline{\mathcal{O}_T})$ is a *viscosity solution* of (5.7) if and only if it is simultaneously a sub- and supersolution of (5.7). If, in addition, $v|_{\{t=T\}} = g$ on $[0,\infty)$, then v is a viscosity solution of the terminal problem (5.7)-(5.8).

Let us now discuss finiteness of the interval terms $\overline{D}, \underline{D}$ in a slightly more general context than the above definition. Here we only require the test function to be continuous on $\overline{\mathcal{O}_T}$ and have at most linear growth. We have three cases to consider:

(i) If $v(t,x) = \phi(t,x) > g(x)$, then the integral is zero for both subsolutions and supersolutions.

(ii) If $v(t,x) = \phi(t,x) < g(x)$, then this together with non-negativity of v implies that $K - x = g(x) > 0$. By continuity $\phi(t,xe^z) - (K - xe^z) < 0$ in a neighborhood of $z = 0$ so the integrand vanishes near the possible singularity of $\nu_t(\cdot)$ for both sub- and supersolutions.

(iii) If $v(t,x) = \phi(t,x) = g(x)$, then the integral vanishes for supersolutions. For subsolutions, we only need to consider the case $x < K$ by the strict positivity assumption in the definition. Then

$$\phi(t,xe^z) - (K - xe^z) = \phi(t,xe^z) - g(xe^z)$$

in a neighborhood of $z = 0$, so the integrand again vanishes.

Remarks 6.4. (i) For the integrand of the operator D we have

$$\begin{aligned}
&H_*(v(t,xe^z) - (K - xe^z))(v(t,xe^z) - (K - xe^z))\\
&= H(v(t,xe^z) - (K - xe^z))(v(t,xe^z) - (K - xe^z))\\
&= (v(t,xe^z) - (K - xe^z))^+.
\end{aligned}$$

In addition, if v is *strictly* positive everywhere in $[0,T) \times (0,+\infty)$ then

$$H_*(v(t,xe^z) - (K - xe^z)) = H_*(v(t,xe^z) - g(xe^z)). \tag{6.5}$$

In principle any of the above expressions could be used in the definition of D. However, since (numerical) approximations may take the value zero at least in some region, we want to allow for this possibility in our definition.

(ii) In recent papers, some other ways for writing the semilinear Black and Scholes equation have appeared, and we should point out the connection of our formulation to these. For this, we define the *free boundary* for the American put option as

$$x_p(t) := \sup\{x : v(t,x) = g(x)\}.$$

From equation (6.5), we have that the integrand of D is nonzero if and only if $z > \log(x_p(t)/x)$. Assuming we have verified the intuition that $v(t,x) = g(x)$ if and only if

$x < x_p(t)$, then the semilinear equation can be written in terms of the free boundary as

$$\begin{aligned} &\mathcal{L}_{BS}v(t,x) + B(t,x,v) \\ &+1_{x<x_p(t)}(x)\Big(rK - dx - \int_{\log(x_p(t)/x)}^{\infty} v(t,xe^z) - (K - xe^z)\nu_t(dz)\Big)^+ = 0. \end{aligned} \tag{6.6}$$

This equation (and its log-transformed version) are used, for example, in [35] and [21]. However, even in the case of the classical Black and Scholes market it is known that the second order derivative with respect to x does not exist at the free boundary, so the equation can not be interpreted in the classical sense on the whole domain. This has not been clearly pointed out in the previous literature, which mainly deals with numerical methods.

At this point, we note the following continuity properties for the integral terms. Let v be a function and $\{(t_k,x_k)\}_{k\geq 1}, (t,x) = (t_0,x_0)$ be points in $\mathcal{O}_T$ such that v is $C_1^{1,2}$ a (t,k) for $k \geq 0$ and $(t_k,x_k) \to (t,x)$ in $\mathcal{O}_T$. Suppose, in addition, that $v \in USC(\mathcal{O}_T)$. By the sublinear growth and a general version of Fatou's lemma (see [5], pages 48 and 295)

$$\limsup_{k\to\infty} B(t_k,x_k,v) \leq B(t,x,v).$$

Similarly, if $v \in LSC(\mathcal{O}_T)$ we attain

$$\liminf_{k\to\infty} B(t_k,x_k,v) \geq B(t,x,v)$$

and for $v \in C(\mathcal{O}_T)$

$$\lim_{k\to\infty} B(t_k,x_k,v) = B(t,x,v).$$

One can similarly verify that the mapping $(t,x) \mapsto \overline{D}(t,x,v)$ is continuous on the relative topology of the set $A := \{g - v \geq 0\} \cap \{v > 0\}$, where we denote

$$\{f \geq 0\} = \{(t,x) | f(t,x) \geq 0\}$$

for a function f. That is, if $(t_k,x_k) \to (t,x)$ in A, then $\overline{D}(t_k,x_k,v) \to \overline{D}(t,x,v)$. In the complement of A, the integral $\overline{D}$ vanishes as does $H^*(g(x) - v(t,x))$. From these facts it follows that q^* is upper semicontinuous. Lower semicontinuity of q_* follows identically.

The next lemma lists some useful properties of the equation and its viscosity solutions. Especially, the monotonicity property of the non-local operators stated in (i) is crucial in many of the proofs that follow. Note that while the integral operator B is clearly monotonically increasing in the non-local part v, the reaction term q is monotonically decreasing in v, so the result is nontrivial. We note here also that the operator is monotonically increasing in $\partial_x^2 v$.

Suppose $v \in C_1(\mathcal{O}_T)$ is $C^{1,2}$ at (t,x). Then we say that v satisfies the subsolution (supersolution) inequality in the *classical* sense at (t,x) if we can replace the test function by v everywhere in the corresponding inequalities (6.2) and (6.3).

Proposition 6.7. *(i) Suppose that v_1, v_2 are continuous functions with at most linear growth which are $C^{1,2}$ at (t,x). If $v_1 - v_2$ has a global minimum equal to zero at (t,x), then*

$$B(t,x,v_1) + q(t,x,v_1) \geq B(t,x,v_2) + q(t,x,v_2), \tag{6.8}$$

where q stands for both q_* (on the set $\{v_2 > 0\}$) and q^*. In addition,

$$B(t,x,v_1 + C) + q(t,x,v_1 + C) = B(t,x,v_1) + q(t,x,v_1) \tag{6.9}$$

for any constant $C \geq 0$.

(ii) Let (t_k, x_k), $k = 1, 2, ...$, and (t, x) be such that $(t_k, x_k) \to (t, x)$ as $k \to \infty$. Suppose that there exists an associated collection of functions $v^{t_k,x_k}, v^{t,x}$ that are $C^{1,2}$ at (t_k, x_k), (t,x), respectively, and

$$\partial_x^n v^{t_k,x_k}(t_k, x_k) \to \partial_x^n v^{t,x}(t, x), \qquad n = 0, 1, 2$$

as $k \to \infty$. Then, on the set $\{v > 0\}$, the function

$$f^* : (t, x) \mapsto B(t, x, v^{t,x}) + q^*(t, x, v^{t,x})$$

satisfies

$$\limsup_{k\to\infty} f^*(t_k, x_k) \le f^*(t, x).$$

Similarly, the function

$$f_* : (t, x) \mapsto B(t, x, v^{t,x}) + q_*(t, x, v^{t,x})$$

satisfies

$$\liminf_{k\to\infty} f_*(t_k, x_k) \le f_*(t, x).$$

Especially, if $v^{t,x} \in C_1(\mathcal{O}_T), (t, x) \in \mathcal{O}_T$ is a continuum of functions, then f^* is lower semicontinuous and f_* is upper semicontinuous in $\mathcal{O}_T$.

(iii) Suppose $v \in C_1(\mathcal{O}_T)$ is $C^{1,2}$ at (t, x), and satisfies the subsolution (supersolution) inequality in the classical sense at (t, x). Then v also satisfies the subsolution (supersolution) inequality in the viscosity sense at (t, x).

(iv) Conversely, suppose v is a subsolution (supersolution) in the viscosity sense, and $\hat{v} \ge v$ ($\hat{v} \le v$) is $C^{1,2}$ at (t, x). Then $\hat{v}$ satisfies the subsolution (supersolution) inequality in the classical sense at (t, x).

(v) If (1.1) has a classical solution $v \in C^{1,2}(\mathcal{O}_T)$, then it is also a viscosity solution.

(vi) Suppose $v \in USC(\mathcal{O}_T)$ ($v \in LSC(\mathcal{O}_T)$) satisfies the supersolution (subsolution) property for $x > 0$. Then v satisfies the supersolution (subsolution) property at $x = 0$ as well.

Proof. To confirm (i), first observe that because $v_1(t, x) = v_2(t, x)$ we have either (I) $q(t, x, v_1) = q(t, x, v_1) = 0$ or (II) $q(t, x, v_1) = c(t, x, v_1)$ and $q(t, x, v_2) = c(t, x, v_1)$. In case (I) the claim holds by monotonicity of the integral term B. For case (II), we have $\underline{D}(t, x, v_i) = \overline{D}(t, x, v_i) = D(t, x, v_i)$ for $i = 1, 2$. From the assumptions we have $v_1(t, x) = v_2(t, x)$, $\partial_x v_1(t, x) = \partial_x v_2(t, x)$, which implies also that

$$B(t, x, v_2) - B(t, x, v_1) = \int_{\mathbb{R}} [v_2(t, xe^z) - v_1(t, xe^z)]\nu_t(dz),$$

and especially that the integral on the right hand side is well defined. Now, notice that an elementary estimation yields that for $f^+(x) := \max\{0, f(x)\}$ we have $f^+(x) - g^+(x) \le (f(x) - g(x))^+$ for any functions f, g. Using this and $H_*(v(t, xe^z) - (K - xe^z))(v(t, xe^z) - (K - xe^z)) = (v(t, xe^z) - (K - xe^z))^+$, we deduce

$$\begin{aligned}
& B(t, x, v_2) + q(t, x, v_2) - B(t, x, v_1) - q(t, x, v_1) \\
= \;& B(t, x, v_2) - B(t, x, v_1) + \Big(rK - dx - D(t, x, v_2)\Big)^+ - \Big(rK - dx - D(t, x, v_1)\Big)^+ \\
\le \;& B(t, x, v_2) - B(t, x, v_1) + \Big(D(t, x, v_1) - D(t, x, v_2)\Big)^+ \\
\le \;& \int_{\mathbb{R}} [v_2(t, xe^z) - v_1(t, xe^z)]\nu_t(dz) + \int_{\mathbb{R}} \Big(v_1(t, xe^z) - v_2(t, xe^z)\Big)^+ \nu_t(dz) \\
= \;& 0,
\end{aligned}$$

where the last equality follows by $v_1 \geq v_2$. To verify (6.9), note that on the one hand, monotonicity of $B+q$ implies

$$B(t,x,v_1+C)+q(t,x,v_1+C) \geq B(t,x,v_1)+q(t,x,v_1).$$

On the other hand,

$$B(t,x,v_1+C)+q(t,x,v_1+C) = B(t,x,v_1)+q(t,x,v_1+C) \leq B(t,x,v_1)+q(t,x,v_1).$$

Next, (ii) follows by continuity of v, the assumptions on the family $v^{t,x}$ and the continuity properties of the integral terms. Claim (iii) follows by standard application of the necessary criteria for maxima of differentiable functions and monotonicity properties of the operator. Claim (v) is a direct consequence of (iii). To prove (iv) for the case of a subsolution, we pick $\bar{\phi} \in C^{1,2}(\mathcal{O}_T)$ such that $\bar{\phi} \geq v$, $v(t,x) = \bar{\phi}(t,x)$, $\partial_t v(t,x) = \partial_t \bar{\phi}(t,x)$, $\partial_x v(t,x) = \partial_x \bar{\phi}(t,x)$, and $\partial_x^2 v(t,x) = \partial_x^2 \bar{\phi}(t,x)$. This can be done by the construction of Evans, see [63, Proposition 4.5.4]. Moreover, let $\{v_k\}_{k=1}^{\infty} \subset C_1^{\infty}(\mathcal{O}_T)$ such that $v_k \downarrow v$ almost everywhere as $k \to \infty$. Let $\mathcal{X}_k$ be a smooth function such that $0 \leq \mathcal{X}_k \leq 1$, $\mathcal{X}_k = 1$ in a ball with radius $1/2k$ and center at (t,x), and $\mathcal{X}_k = 0$ outside a ball with radius $1/k$ and center at (t,x). Then

$$\phi_k(s,y) := \mathcal{X}_k(y)\bar{\phi}(s,y) + (1-\mathcal{X}_k(y))v_k(s,y)$$

defines a sequence of test functions such that

$$\partial_x^n \phi_k(t,x) = \partial_x^n v(t,x), \qquad n = 0,1,2$$

and $\phi^k \downarrow v$ everywhere as $k \to \infty$. Note especially that by monotone convergence

$$\lim_{k\to\infty} B(t,x,\phi_k) = B(t,x,v),$$

so the sequence of integrals has a well defined limit. The claim then follows from (ii), and proof for the case of a supersolution is symmetric. Claim (vi) can be seen to hold by adapting arguments in Lemma 4.1 of [16] and using the semicontinuity of the functions f^*, f_* above. ☐

7. Existence

In this section, we show that the value function (4.1) is a viscosity solution of (5.7)-(5.8), thereby providing the existence result. We repeat that we only need the continuity and linear growth properties of the American option value.

For the existence result, the following lemma will be useful.

Lemma 7.1. *The pay-off function g is a viscosity subsolution of the semilinear Black and Scholes equation (1.1).*

Proof. We prove the lemma for the put option $g(x) = (K-x)^+$, the proof for the call option is similar. We will show that, in fact, g satisfies the equation in the classical sense whenever $x \neq K$. Furthermore, if $x = K$ there is no smooth function $\phi \in C^{1,2}(\mathcal{O}_T)$ such that $\phi \geq g$, $\phi(t,K) = g(K)$. Then the claim follows by proposition 6.7.

Let $x \neq K$ and note that $q^*(t,x,g) = c(t,x,g)$, $\overline{D}(t,x,g) = 0$ everywhere. We have four cases to consider. If $x < K$ and $rK - dx \geq 0$ (Case I), we have

$$\mathcal{L}_{BS} g(x) + B(t,x,g) + q^*(t,x,g) = 0.$$

If $x > K$ and $rK - dx \leq 0$ (Case II), then $g = 0$ in a neighborhood of x, so the claim holds trivially. If $x < K$ and $rK - dx \leq 0$ (Case III), then

$$\mathcal{L}_{BS} g(x) + B(t,x,g) + q^*(t,x,g) = -(rK-dx) \geq 0.$$

If $x > K$ and $rK - dx \geq 0$ (Case IV), then

$$\mathcal{L}_{BS}g(x) + B(t,x,g) + q^*(t,x,g) = \int_{\mathbb{R}} g(xe^z)\nu_t(dz) + rK - dx \geq 0.$$

□

The following lemma states that the linear part of equation (5.7) comes from the characteristic operator of the space-time process $(u, S(u))$.

Lemma 7.2. *For $n \in \mathbb{N}$, let θ_n be the exit time for the space-time process $(u, S(u))$, $u \in [0,T]$ from a ball with radius $1/n$ and center at (t,x). Then, for $\phi \in C_1^{1,2}(\overline{\mathcal{O}_T})$,*

$$\frac{\mathbb{E}^{t,x}[e^{-r(\theta_n - t)}\phi(\theta_n, S(\theta_n))] - \phi(t,x)}{\mathbb{E}^{t,x}[\theta_n] - t} \to \mathcal{L}_{BS}\phi(t,x) + B(t,x,\phi) \tag{7.3}$$

as $n \to \infty$.

Proof. Let us begin by verifying that, for $\phi \in C_1^{1,2}(\overline{\mathcal{O}_T})$, the Dynkin formula

$$\begin{aligned} &\mathbb{E}^{t,x}[e^{-r(\theta - t)}\phi(\theta, S(\theta))] \\ &= \phi(t,x) + \mathbb{E}^{t,x}\Big[\int_t^{\theta_n} e^{-r(u-t)}\big(\mathcal{L}_{BS}\phi(u,S(u)) + B(u,S(u),\phi)\big)du\Big] \end{aligned}$$

holds in this case. First note that the left hand side is finite because of (4.3) and the right hand side is finite by the definition of θ_n and continuity of the integrand. The claim follows by Itô's formula for semimartingales and the fact that the stochastic integrals have zero expectation because of the localization with θ_n. Using the Dynkin formula above,

$$\begin{aligned} &\Big|\frac{\mathbb{E}^{t,x}[e^{-r(\theta_n - t)}\phi(\theta_n, S(\theta_n))] - \phi(t,x)}{\mathbb{E}^{t,x}[\theta_n] - t} - (\mathcal{L}_{BS}\phi(t,x) + B(t,x,\phi))\Big| \\ = &\frac{\Big|\mathbb{E}^{t,x}\big[\int_t^{\theta_n} e^{-r(u-t)}\big(\mathcal{L}_{BS}\phi(u,S(u)) + B(u,S(u),\phi) - \mathcal{L}_{BS}\phi(t,x) - B(t,x,\phi)\big)du\big]\Big|}{|\mathbb{E}^{t,x}[\theta_n] - t|} \\ \leq &\sup_{(u,y)\in B_{1/n}(t,x)} \Big|\mathcal{L}_{BS}\phi(u,y) + B(u,y,\phi) - \mathcal{L}_{BS}\phi(t,x) - B(t,x,\phi)\Big| \to 0 \end{aligned}$$

as $n \to \infty$, where in the last inequality we have again used the definition of θ_n.

□

The next theorem shows that the value function for the American put option is a viscosity solution of the semilinear Black and Scholes partial integro-differential equation. The proof is inspired by the formal derivation of (5.7) given in Section 5.

Theorem 7.4. *The value function $v(t,x)$ defined in (4.1) is a viscosity solution of the terminal value problem (5.7)-(5.8).*

Proof. Continuity of the value function follows from Proposition 4.7, and it is clear from the definition that the value function satisfies the terminal condition. It remains to prove that v is a subsolution and a supersolution of the semilinear equation (5.7).

We prove first the supersolution property. Let $\phi \in C_1^{1,2}(\overline{\mathcal{O}_T})$ and $(t,x) \in \mathcal{O}_T$ be such that $v \geq \phi$ and $v(t,x) = \phi(t,x)$. For $n \in \mathbb{N}$, let θ_n be the exit time of the space-time process $(u, S(u))$ from a ball with radius $1/n$ and center at (t,x). Using $v \geq \phi$, $v(t,x) = g(x)$ together with the first dynamic programming inequality (4.5) and Lemma

7.2, we deduce

$$\begin{aligned}
0 &= \frac{v(t,x)-\phi(t,x)}{\mathbb{E}^{t,x}[\theta_n]-t} \\
&\geq \frac{\mathbb{E}^{t,x}[e^{-r(\theta_n-t)}v(\theta_n,S(\theta_n))]-\phi(t,x)}{\mathbb{E}^{t,x}[\theta_n]-t} \\
&\geq \frac{\mathbb{E}^{t,x}[e^{-r(\theta_n-t)}\phi(\theta_n,S(\theta_n))]-\phi(t,x)}{\mathbb{E}^{t,x}[\theta_n]-t} \\
&\to \mathcal{L}_{BS}\phi(t,x)+B(t,x,\phi),
\end{aligned}$$

as $n\to\infty$. Notice also that $\phi(t,x)=v(t,x)\geq g(x)$ so $q_*(t,x,\phi)=0$ in $[0,T)\times\mathbb{R}_+$. Thus,

$$\mathcal{L}_{BS}\phi(t,x)+B(t,x,\phi)+q_*(t,x,\phi)\leq 0,$$

so v is a supersolution of the semilinear Black and Scholes equation (5.7).

Let us next prove that v is a subsolution. Let $\phi\in C_1^{1,2}(\overline{\mathcal{O}_T})$ and $(t,x)\in\mathcal{O}_T$ be such that $v\leq\phi$ and $v(t,x)=\phi(t,x)$. Assume first that (t,x) is in the continuation region, i.e. $v(t,x)>g(x)$. Then $q^*(t,x,\phi)=0$, and similarly as above, (4.6) implies

$$\mathcal{L}_{BS}\phi(t,x)+B(t,x,\phi)\geq 0.$$

Assume finally that (t,x) is in the stopping region (so that $\phi(t,x)=v(t,x)=g(x)$). Since $\phi(t,x)=v(t,x)=g(x)$ and $\phi\geq v\geq g$, we conclude by the subsolution property of g (Lemma 7.1) that

$$\mathcal{L}_{BS}\phi(t,x)+B(t,x,\phi)+q^*(t,x,\phi)\geq 0,$$

which verifies the subsolution property of v, concluding our proof. □

Remark 7.5. Notice that the above proof applies to both the call and the put option, once it is recognized that the corresponding payoff g is a subsolution of the given equation in both cases. The introduction of the localizing stopping time θ_n is necessary for call options, while for put options one could work out a simpler proof using compactly supported test functions and the infinitesimal generator of $(u,S^{t,x}(u))$.

8. A strong comparison principle

In this section, we follow a quite self-contained approach outlined in [63] for proving comparison principles. To adapt this approach to our partial integro-differential equation, we also borrow ideas from [8], [16], [40] and [56].

As mentioned in the introduction, Jakobsen and Karlsen discuss in [40] some issues concerning the applicability of Ishii's lemma in connection with integro-differential equations. We note that their results are not applicable as such here because of the discontinuity in the reaction term. The subsequent work of Arisawa [4], [3], and the recent paper by Barles and Imbert [9] also apply maximum principles. Rather than reworking through the rather long proofs of these types of abstract maximum principles, we work with the approximative methods that are the main tools behind such results (see [28]) and allow for a rather direct proof. We mention that insight gained in this way is used in [62] to show how Ishii's lemma can in fact be applied for integro-differential equations if this is done in a careful manner.

We will construct approximations of the sub- and supersolutions which are then sub- and supersolutions of an approximate semilinear Black and Scholes equation. We begin by introducing the approximations, which have enough regularity to allow for differentiation almost anywhere.

Definition 8.1. Let $v \in USC(\overline{\mathcal{O}_T})$ satisfy $v(t,x) \leq L(1+x)$ in $\overline{\mathcal{O}_T}$ and let $\gamma < \frac{1}{2\sqrt{L}}$. The *sup convolution* v^γ is defined as

$$v^\gamma(t,x) = \sup_{(s,y)\in\overline{\mathcal{O}_T}} \Big(v(s,y) - \frac{1}{2\gamma^2}[|t-s|^2 + |x-y|^2] \Big).$$

Let $v \in LSC(\overline{\mathcal{O}_T})$ satisfy $v(t,x) \geq -L(1+x)$ in $\overline{\mathcal{O}_T}$ and let $\gamma < \frac{1}{2\sqrt{L}}$. The *inf convolution* v_γ is defined as

$$v_\gamma(t,x) = \inf_{(s,y)\in\overline{\mathcal{O}_T}} \Big(u(s,y) + \frac{1}{2\gamma^2}[|t-s|^2 + |x-y|^2] \Big).$$

These approximations have some nice properties that will be useful later. The following lemma lists some of these.

Lemma 8.2. *(i) Let $v \in USC(\overline{\mathcal{O}_T})$ satisfy $0 \leq v(t,x) \leq L(1+x)$ in $\overline{\mathcal{O}_T}$ and let $0 < \gamma < \frac{1}{2L^{1/2}}$. Then $0 \leq v^\gamma(t,x) \leq 2L(1+x)$ and $v^\gamma(t,x) + \frac{1}{2\gamma^2}(t^2+x^2)$ is convex (i.e. v^γ is semiconvex). Define $C(t,x) := (4L(1+x) - 2v(t,x))^{1/2}$. If $(t,x) \in \mathcal{O}_T$ is such that $\operatorname{dist}((t,x), \partial\mathcal{O}_T) > \sqrt{2}\gamma C(t,x)$, then there exists $(\hat{t},\hat{x})$ such that $|(t,x) - (\hat{t},\hat{x})| \leq \sqrt{2}\gamma C(t,x)$ and*

$$v^\gamma(t,x) = v(\hat{t},\hat{x}) - \frac{1}{2\gamma^2}[|t-\hat{t}|^2 + |x-\hat{x}|^2]. \tag{8.3}$$

(ii) Let $v \in LSC(\overline{\mathcal{O}_T})$ satisfy $0 \leq v(t,x) \leq C(1+x)$ in $\overline{\mathcal{O}_T}$ and let $0 < \gamma < \frac{1}{2L^{1/2}}$. Then $0 \leq v_\gamma(t,x) \leq 2L(1+x)$ and $v_\gamma(t,x) - \frac{1}{2\gamma^2}(t^2+x^2)$ is concave (i.e. v_γ is semiconcave). If $(t,x) \in \mathcal{O}_T$ is such that $\operatorname{dist}((t,x), \partial\mathcal{O}_T) > \sqrt{2}\gamma C(t,x)$, then there exists $(\hat{t},\hat{x})$ such that $|(t,x) - (\hat{t},\hat{x})| \leq \sqrt{2}\gamma C(t,x)$ and

$$v_\gamma(t,x) = v(\hat{t},\hat{x}) + \frac{1}{2\gamma^2}[|t-\hat{t}|^2 + |x-\hat{x}|^2]. \tag{8.4}$$

(iii) Finally, we have that $v^\gamma \downarrow v$ and $v_\gamma \uparrow v$ pointwise as $\gamma \to 0$.

Especially, from Alexandrov's theorem it follows that the sup and inf convolutions are $C^{1,2}$ almost everywhere. We refer to [6, 32, 63] for this fact and proofs of results like those in lemma 8.2.

Let G denote the Hamiltonian

$$G^*_{(*)}(t,x,q,p,P,v) = rxp + \frac{1}{2}\sigma^2(t)x^2P - rq + B(t,x,v) + q^*_{(*)}(t,x,v)$$

Now, if v is a function satisfying assumptions of Lemma (8.2), we define for fixed γ

$$\mathcal{O}_T^{v,\gamma} := \Big\{ (t,x) \in \mathcal{O}_T \Big| \operatorname{dist}((t,x), \partial\mathcal{O}_T) > \sqrt{2}\gamma C(t,x) \Big\},$$

where $C(t,x)$ is defined as in Lemma (8.2). Moreover, let τ_k^l denote the shift operator defined by

$$\tau_k^l \phi(t,x) := \phi(t+k, x+l)$$

for any function ϕ and any $(t,x), (t+h, x+h)$ in the domain of definition of ϕ. To introduce suitable approximations of the semilinear Black and Scholes equation we will need the operators

$$\begin{aligned} G^\gamma(t,x,q,p,P,v) = \sup_{(s,y)} \Big\{ & ryp + \frac{1}{2}\sigma^2(s)y^2P - rq + B(s,y,\tau_{t-s}^{x-y}v) \\ & + q^*(s,y,\tau_{t-s}^{x-y}v) \Big| |(t,x)-(s,y)| \leq \sqrt{2}\gamma C(t,x) \Big\} \end{aligned} \tag{8.5}$$

and

$$G_\gamma(t,x,q,p,P,v) = \inf_{(s,y)} \Big\{ ryp + \frac{1}{2}\sigma^2(s)y^2P - rq + B(s,y,\tau^{x-y}_{t-s}v) + q_*(s,y,\tau^{x-y}_{t-s}v) \Big| |(t,x)-(s,y)| \le \sqrt{2}\gamma C(t,x) \Big\}. \tag{8.6}$$

Notice that both G^γ and G_γ inherit the same monotonicity in q, P, and v as the operator $\mathcal{L}$ (see inequality (6.8)).

The following lemma shows that the semiconvex approximations of sub- and supersolutions satisfy an equation modified by the above operators.

Lemma 8.7. *(a) Suppose $v \in USC(\overline{\mathcal{O}_T})$, $0 \le v(t,x) \le L(1+x)$ is a subsolution of the semilinear equation (5.7) and v^γ is the sup convolution of v for $0 < \gamma < \frac{1}{2L^{1/2}}$. If v^γ is $C^{1,2}$ at (t,x), then*

$$\partial_t v^\gamma(t,x) + G^\gamma(t,x,v^\gamma(t,x),\partial_x v^\gamma(t,x),\partial_x^2 v^\gamma(t,x),v^\gamma) \ge 0 \quad \text{in } \mathcal{O}_T^{v,\gamma} \cap \{v^\gamma > 0\}. \tag{8.8}$$

(b) Suppose $v \in LSC(\overline{\mathcal{O}_T})$, $0 \le v \le L(1+x)$ is a viscosity supersolution of the semilinear equation (5.7) and v_γ is the inf convolution of v for $0 < \gamma < \frac{1}{2L^{1/2}}$. If v_γ is $C^{1,2}$ at (t,x), then

$$\partial_t v_\gamma(t,x) + G_\gamma(t,x,v_\gamma(t,x),\partial_x v_\gamma(t,x),\partial_x^2 v_\gamma(t,x),v_\gamma) \le 0 \quad \text{in } \mathcal{O}_T^{v,\gamma}. \tag{8.9}$$

Proof. We give the proof for (a), the proof of (b) is similar. Suppose v^γ is $C^{1,2}$ at $(\bar t,\bar x) \in \mathcal{O}_T^{v,\gamma}$. Suppose that $(\hat t,\hat x)$ satisfies (8.3). For any (t,x), we have

$$\begin{aligned} v(s,y) - \frac{1}{2\gamma^2}[|t-s|^2 + |x-y|^2] - v(t,x) \quad &\le 0 \\ &= v^\gamma(\bar t,\bar x) - v^\gamma(\bar t,\bar x) \\ &= v(\hat t,\hat x) - \frac{1}{2\gamma^2}[|\bar t-\hat t|^2 + |\bar x-\hat x|^2] - v^\gamma(\bar t,\bar x). \end{aligned}$$

Choosing $(t,x) = (s-\hat t+\bar t, y-\hat x+\bar x)$ in this inequality it follows that the mapping

$$v(s,y) - v^\gamma(s-\hat t+\bar t, y-\hat x+\bar x)$$

attains a global maximum at $(\hat t,\hat x)$. Let us define a function

$$\hat v(s,y) := v^\gamma(s-\hat t+\bar t, y-\hat x+\bar x) + v(\hat t,\hat x) - v^\gamma(\bar t,\bar x).$$

Since $(\bar t,\bar x) \in \mathcal{O}_T^{v,\gamma}$ and $|(\bar t,\bar x) - (\hat t,\hat x)| \le \sqrt{2}\gamma C(\bar t,\bar x)$, we have that the function is well defined for all $y > 0$ and s in a neighborhood of $\hat t$. By the above estimates, we have that $\hat v \ge v$ with $\hat v(\hat t,\hat x) = v(\hat t,\hat x)$. Furthermore, $\hat v$ is differentiable at $(\hat t,\hat x)$ with $\partial_t \hat v(\hat t,\hat x) = \partial_t v^\gamma(\bar t,\bar x)$, $\partial_x \hat v(\hat t,\hat x) = \partial_x v^\gamma(\bar t,\bar x)$ and $\partial_x^2 \hat v(\hat t,\hat x) = \partial_x^2 v^\gamma(\bar t,\bar x)$. Thus, by Proposition 6.7, (iii) $\hat v$ satisfies the subsolution inequality in the classical sense so that

$$\begin{aligned} &\partial_t v^\gamma(\bar t,\bar x) + (r-d)\hat x \partial_x v^\gamma(\bar t,\bar x) + \frac{1}{2}\sigma^2(\hat t)\hat x^2 \partial_x^2 v^\gamma(\bar t,\bar x) - r v^\gamma(\bar t,\bar x) + B(\hat t,\hat x,\hat v) + q^*(\hat t,\hat x,\hat v) \\ =&\partial_t \hat v(\hat t,\hat x) + (r-d)\hat x \partial_x \hat v(\hat t,\hat x) + \frac{1}{2}\sigma^2(\hat t)\hat x^2 \partial_x^2 \hat v(\hat t,\hat x) - r\hat v(\hat t,\hat x) + B(\hat t,\hat x,\hat v) + q^*(\hat t,\hat x,\hat v) \ge 0, \end{aligned}$$

and by definition of G^γ, we obtain

$$\partial_t v^\gamma(\bar t,\bar x) + G^\gamma(\bar t,\bar x,v^\gamma(\bar t,\bar x),\partial_x v^\gamma(\bar t,\bar x),\partial_x^2 v^\gamma(\bar t,\bar x),v^\gamma) \ge 0.$$

where we have used (6.9) with $C = v(\hat t,\hat x) - v^\gamma(\bar t,\bar x) \ge 0$. This completes the proof. $\square$

Our second main result is a comparison principle for the terminal value problem (5.7)-(5.8). The comparison principle is strong in the sense that it applies for a class of semi-continuous functions satisfying a natural growth condition. Besides implying uniqueness of the viscosity solution, the comparison principle is useful in proving convergence of approximate solutions to the equation. Here we apply the approximation procedures given in this section and which are at the heart of the more abstract maximum principles. The proof depends fundamentally on the monotonicity property of the whole non-local part

$$v \mapsto B(t,x,v) + q(t,x,v)$$

of the operator.

Theorem 8.10 (Comparison principle)**.** *Suppose $\underline{v} \in USC_1^+(\overline{\mathcal{O}_T})$ is a subsolution and $\overline{v} \in LSC_1^+(\overline{\mathcal{O}_T})$ is a supersolution of the semilinear BS equation, satisfying*

$$\underline{v}(T,x) \le \overline{v}(T,x), \qquad x \in [0,\infty). \tag{8.11}$$

Then

$$\underline{v} \le \overline{v} \quad \text{on } [0,T] \times \mathbb{R}_+. \tag{8.12}$$

Proof. Let $\mu > 0$, define $\overline{v}^\mu(t,x) := \overline{v}(t,x) + \mu(T-t)$. We prove the claim holds for $\overline{v}^\mu$, and the main claim follows by taking $\mu \to 0$.

As noted before, we assume first the claim does not hold and then proceed to derive a contradiction from this. Thus, suppose there exists $\delta > 0$ and $(\bar{t},\overline{x}) \in [0,T] \times \mathbb{R}_+$ such that

$$\underline{v}(\bar{t},\overline{x}) \ge \overline{v}^\mu(\bar{t},\overline{x}) + 3\delta. \tag{8.13}$$

Let $\underline{v}^\gamma, \overline{v}_\gamma$ denote the sup and inf convolutions of $\underline{v}$ and $\overline{v}$, respectively. Furthermore, let $\overline{v}^\mu_\gamma(t,x) := \overline{v}_\gamma(t,x) + \mu(T-t)$. Then, since $\underline{v}^\gamma \downarrow \underline{v}$ and $\overline{v}^\mu_\gamma \uparrow \overline{v}^\mu$ as $\gamma \downarrow 0$, for $\gamma > 0$ we have that the functions $\underline{v}^\gamma$ and $\overline{v}^\mu_\gamma$ satisfy

$$\underline{v}^\gamma(\bar{t},\overline{x}) - \overline{v}^\mu_\gamma(\bar{t},\overline{x}) \ge \underline{v}(\bar{t},\overline{x}) - \overline{v}^\mu(\bar{t},\overline{x}) \ge 3\delta,$$

so

$$\underline{v}^\gamma(\bar{t},\overline{x}) \ge \overline{v}^\mu_\gamma(\bar{t},\overline{x}) + 3\delta. \tag{8.14}$$

also. We define

$$\Phi(t,x,y) := \underline{v}(t,x) - \overline{v}^\mu(t,y) - \psi(t,x,y), \quad (t,x,y) \in [0,T] \times \mathbb{R}_+ \times \mathbb{R}_+,$$

and

$$\Phi_\gamma(t,x,y) := \underline{v}^\gamma(t,x) - \overline{v}^\mu_\gamma(t,y) - \psi(t,x,y), \quad (t,x,y) \in [0,T] \times \mathbb{R}_+ \times \mathbb{R}_+,$$

for any $\gamma > 0$ where

$$\psi(t,x,y) = \frac{\alpha}{2}(x-y)^2 + \frac{\epsilon}{2} e^{\lambda(T-t)}(x^2+y^2).$$

We note also that $\Phi_0(t,x,y) := \lim_{\gamma\to 0} \Phi_\gamma(t,x,y) = \Phi$. It is standard in viscosity solutions theory [29] to see that (for each fixed γ, ϵ) there exists a sequence of maxima $(t_\alpha, x_\alpha, y_\alpha)$ that converge to a limit point $(t_\epsilon, x_\epsilon, y_\epsilon)$. Furthermore, the maxima $(t_\alpha, x_\alpha, y_\alpha)$ satisfy

$$x_\alpha - y_\alpha \to 0$$

and

$$\alpha|x_\alpha - y_\alpha|^2 \to 0$$

as $\alpha \to \infty$. Note that we have dropped the dependency on ϵ and γ for notational convenience.

Observe that

$$\Phi_\gamma(t_\alpha, x_\alpha, y_\alpha) \geq \Phi_\gamma(\bar{t}, \overline{x}, \overline{x}) = \underline{v}^\gamma(\bar{t}, \overline{x}) - \overline{v}^\mu_\gamma(\bar{t}, \overline{x}) - \epsilon e^{\lambda(T-\bar{t})}\overline{x}^2 \geq 2\delta > 0, \tag{8.15}$$

for any $\alpha > 1$,$\gamma \geq 0$, and any $\epsilon > 0$ that is small enough. This implies

$$\underline{v}^\gamma(t_\alpha, x_\alpha) \geq \overline{v}^\mu_\gamma(t_\alpha, y_\alpha) + 2\delta > 0 \tag{8.16}$$

for any $\alpha > 1$, $\gamma \geq 0$ and any ϵ sufficiently small.

Let us now look at the special case $t_\epsilon = T$. Note that

$$\underline{v}(\bar{t}, \overline{x}) - \overline{v}^\mu(\bar{t}, \overline{x}) - \epsilon e^{\lambda(T-\bar{t})}\overline{x}^2 \leq \Phi(t_\alpha, x_\alpha, y_\alpha) \leq \underline{v}(t_\alpha, x_\alpha) - \overline{v}^\mu(t_\alpha, y_\alpha).$$

By the upper semicontinuity of $\underline{v}$, $-\overline{v}^\mu$ and since $\underline{v}|_{t=T} \leq \overline{v}^\mu_{t=T}$ on $[0, \infty)$, we can send $\alpha \uparrow \infty$ in this inequality to obtain $\underline{v}(\bar{t}, \overline{x}) - \overline{v}^\mu(\bar{t}, \overline{x}) - \epsilon e^{\lambda(T-\bar{t})}\overline{x}^2 \leq 0$, which contradicts (8.15). Hence we may assume from now on that $t_\epsilon < T$. Then $t_\alpha < T$ for any α sufficiently large.

Let Q^ϵ be a compact and convex set in $\overline{\mathcal{O}_T}$ such that the subsequence of maximum points $(t_\alpha, x_\alpha, y_\alpha)$ is contained in Q^ϵ for $\alpha > 1, 0 < \gamma < 1/2$. Then the restriction of ψ to Q^ϵ is smooth with bounded derivatives, which implies its semiconcavity. Thus Φ is semiconvex on Q. Consequently, for small $\gamma > 0$,

$$\hat{\Phi}_\gamma(t, x, y) := \Phi_\gamma(t, x, y) - s(|t - t_\alpha|^2 + |x - x_\alpha|^2 + |y - y_\alpha|^2)$$

is semiconvex on Q^ϵ and attains a strict maximum at $(t_\alpha, x_\alpha, y_\alpha)$. By the Lemmas of Alexandrov and Jensen (see p. 202 in [63]), there exist $q, p, \hat{p} \in \mathbb{R}$ (depending on $\gamma > 0$) with

$$|q| + |p| + |\hat{p}| \leq \gamma, \tag{8.17}$$

and $(\hat{t}_\alpha, \hat{x}_\alpha, \hat{y}_\alpha)$ with

$$|\hat{t}_\alpha - t_\alpha| + |\hat{x}_\alpha - x_\alpha| + |\hat{y}_\alpha - y_\alpha| \leq \gamma, \tag{8.18}$$

such that

$$\hat{\Phi}_\gamma(t, x, y) + qt + px + \hat{p}y \tag{8.19}$$

attains a maximum at $(\hat{t}_\alpha, \hat{x}_\alpha, \hat{y}_\alpha)$, at which $\underline{v}^\gamma(t, x) - \overline{v}^\mu_\gamma(t, y)$ is twice differentiable. By the first- and second-order necessary conditions for a maximum point we must have that

$$\begin{aligned}
\partial_t \underline{v}^\gamma(\hat{t}_\alpha, \hat{x}_\alpha) - \partial_t \overline{v}^\mu_\gamma(\hat{t}_\alpha, \hat{y}_\alpha) &= -\lambda\frac{\epsilon}{2}e^{\lambda(T-\hat{t}_\alpha)}(\hat{x}_\alpha^2 + \hat{y}_\alpha^2) + 2\gamma(\hat{t}_\alpha - t_\alpha) - q, \\
\partial_x \underline{v}^\gamma(\hat{t}_\alpha, \hat{x}_\alpha) &= \alpha(\hat{x}_\alpha - \hat{y}_\alpha) + \epsilon e^{\lambda(T-\hat{t}_\alpha)}\hat{x}_\alpha + 2\gamma(\hat{x}_\alpha - x_\alpha) - p, \\
\partial_y \overline{v}^\mu_\gamma(\hat{t}_\alpha, \hat{y}_\alpha) &= -\alpha(\hat{x}_\alpha - \hat{y}_\alpha) - \epsilon e^{\lambda(T-\hat{t}_\alpha)}\hat{y}_\alpha - 2\gamma(\hat{y}_\alpha - y_\alpha) + \hat{p},
\end{aligned}$$

and

$$\begin{pmatrix} \partial_x^2 \underline{v}^\gamma(\hat{t}_\alpha, \hat{x}_\alpha) & 0 \\ 0 & -\partial_y^2 \overline{v}^\mu_\gamma(\hat{t}_\alpha, \hat{y}_\alpha) \end{pmatrix} \leq \alpha \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} + \Big(\epsilon e^{\lambda(T-\hat{t}_\alpha)} + 2\gamma\Big) I,$$

where I is the 2×2 identity matrix. We also have by Lemma 8.7 that

$$\partial_t \underline{v}^\gamma(\hat{t}_\alpha, \hat{x}_\alpha) + G^\gamma(\hat{t}_\alpha, \hat{x}_\alpha, \underline{v}^\gamma(\hat{t}_\alpha, \hat{x}_\alpha), \partial_x \underline{v}^\gamma(\hat{t}_\alpha, \hat{x}_\alpha), \partial_x^2 \underline{v}^\gamma(\hat{t}_\alpha, \hat{x}_\alpha), \underline{v}^\gamma) \geq 0.$$

and, using monotonicity of G_γ, it is straightforward to notice that $\overline{v}^\mu_\gamma$ satisfies

$$\partial_t \overline{v}^\mu_\gamma(\hat{t}_\alpha, \hat{y}_\alpha) + G_\gamma(\hat{t}_\alpha, \hat{y}_\alpha, \overline{v}^\mu_\gamma(\hat{t}_\alpha, \hat{y}_\alpha), \partial_x \overline{v}^\mu_\gamma(\hat{t}_\alpha, \hat{y}_\alpha), \partial_x^2 \overline{v}^\mu_\gamma(\hat{t}_\alpha, \hat{y}_\alpha), \overline{v}^\mu_\gamma) \leq -\mu.$$

By the definition of G^γ and G_γ, we can find a point $(\bar{t}_\alpha, \bar{x}_\alpha, \bar{y}_\alpha)$ with

$$|\bar{t}_\alpha - \hat{t}_\alpha| + |\bar{x}_\alpha - \hat{x}_\alpha| + |\bar{y}_\alpha - \hat{y}_\alpha| \leq K_\epsilon \gamma^2$$

for some constant K_ϵ only depending on ϵ, such that

$$\begin{aligned}
\mu &\leq \partial_t \underline{v}^\gamma(\hat{t}_\alpha,\hat{x}_\alpha) + G^\gamma(\hat{t}_\alpha,\hat{x}_\alpha,\underline{v}^\gamma(\hat{t}_\alpha,\hat{x}_\alpha),\partial_x\underline{v}^\gamma(\hat{t}_\alpha,\hat{x}_\alpha),\partial_x^2\underline{v}^\gamma(\hat{t}_\alpha,\hat{x}_\alpha),\underline{v}^\gamma)\\
&\quad -\partial_t\overline{v}^\mu_\gamma(\hat{t}_\alpha,\hat{y}_\alpha) - G_\gamma(\hat{t}_\alpha,\hat{y}_\alpha,\overline{v}^\mu_\gamma(\hat{t}_\alpha,\hat{y}_\alpha),\partial_y\overline{v}^\mu_\gamma(\hat{t}_\alpha,\hat{y}_\alpha),\partial_y^2\overline{v}^\mu_\gamma(\hat{t}_\alpha,\hat{y}_\alpha),\overline{v}^\mu_\gamma)\\
&= \partial_t \underline{v}^\gamma(\hat{t}_\alpha,\hat{x}_\alpha) + G^*(\bar{t}_\alpha,\bar{x}_\alpha,\underline{v}^\gamma(\hat{t}_\alpha,\hat{x}_\alpha),\partial_x\underline{v}^\gamma(\hat{t}_\alpha,\hat{x}_\alpha),\partial_x^2\underline{v}^\gamma(\hat{t}_\alpha,\hat{x}_\alpha),\tau^{\hat{x}_\alpha-\bar{x}}_{\hat{t}_\alpha-\bar{t}}\underline{v}^\gamma)\\
&\quad -\partial_t\overline{v}^\mu_\gamma(\hat{t}_\alpha,\hat{y}_\alpha) - G_*(\bar{t}_\alpha,\bar{y}_\alpha,\overline{v}^\mu_\gamma(\hat{t}_\alpha,\hat{y}_\alpha),\partial_y\overline{v}^\mu_\gamma(\hat{t}_\alpha,\hat{y}_\alpha),\partial_y^2\overline{v}^\mu_\gamma(\hat{t}_\alpha,\hat{y}_\alpha),\tau^{\hat{y}_\alpha-\bar{y}}_{\hat{t}_\alpha-\bar{t}}\overline{v}^\mu_\gamma)\\
&= I_1+I_2+I_3+I_4+I_5+I_6,
\end{aligned}$$

where

$$\begin{aligned}
I_1 &= \partial_t\underline{v}^\gamma(\hat{t}_\alpha,\hat{x}_\alpha) - \partial_t\overline{v}^\mu_\gamma(\hat{t}_\alpha,\hat{y}_\alpha),\\
I_2 &= (r-d)[\overline{x}_\alpha\partial_x\underline{v}^\gamma(\hat{t}_\alpha,\hat{x}_\alpha) - \overline{y}_\alpha\partial_y\overline{v}^\mu_\gamma(\hat{t}_\alpha,\hat{y}_\alpha)],\\
I_3 &= \frac{1}{2}\sigma^2(\bar{t}_\alpha)[\overline{x}^2_\alpha\partial_x^2\underline{v}^\gamma(\hat{t}_\alpha,\hat{x}_\alpha) - \overline{y}^2_\alpha\partial_y^2\overline{v}^\mu_\gamma(\hat{t}_\alpha,\hat{y}_\alpha)],\\
I_4 &= r[\overline{v}^\mu_\gamma(\hat{t}_\alpha,\hat{y}_\alpha) - \underline{v}^\gamma(\hat{t}_\alpha,\hat{x}_\alpha)],\\
I_5 &= B(\bar{t}_\alpha,\bar{x}_\alpha,\tau^{\hat{x}_\alpha-\bar{x}_\alpha}_{\hat{t}_\alpha-\bar{t}_\alpha}\underline{v}^\gamma) - B(\bar{t}_\alpha,\bar{y}_\alpha,\tau^{\hat{y}_\alpha-\bar{y}_\alpha}_{\hat{t}_\alpha-\bar{t}_\alpha}\overline{v}^\mu_\gamma),\\
I_6 &= q^*(\bar{t}_\alpha,\bar{x}_\alpha,\tau^{\hat{x}_\alpha-\bar{x}_\alpha}_{\hat{t}_\alpha-\bar{t}_\alpha}\underline{v}^\gamma) - q_*(\bar{t}_\alpha,\bar{y}_\alpha,\tau^{\hat{y}_\alpha-\bar{y}_\alpha}_{\hat{t}_\alpha-\bar{t}_\alpha}\overline{v}^\mu_\gamma).
\end{aligned}$$

We now make estimates, first for the local terms $I_1 - I_4$ and then for the non-local ones I_5, I_6. For the local part, the only non-trivial term is the second order term I_3. For this, notice that fully utilizing the information given by the Jacobian leads to the estimate

$$\begin{aligned}
I_3 &= \frac{1}{2}\sigma^2(\bar{t}_\alpha)\Big\{\begin{pmatrix}\overline{x}_\alpha & \overline{y}_\alpha\end{pmatrix}\begin{pmatrix}\partial_x^2\underline{v}^\gamma(\hat{t}_\alpha,\hat{x}_\alpha) & 0\\ 0 & -\partial_y^2\overline{v}^\mu_\gamma(\hat{t}_\alpha,\hat{y}_\alpha)\end{pmatrix}\begin{pmatrix}\overline{x}_\alpha\\ \overline{y}_\alpha\end{pmatrix}\Big\}\\
&\leq \frac{1}{2}\sigma^2(\bar{t}_\alpha)\Big\{\begin{pmatrix}\overline{x}_\alpha & \overline{y}_\alpha\end{pmatrix}\Big(\alpha\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix} + \epsilon e^{\lambda(T-\hat{t}_\alpha)}I_2\Big)\begin{pmatrix}\overline{x}_\alpha\\ \overline{y}_\alpha\end{pmatrix} + 2\gamma(\overline{x}^2_\alpha - \overline{y}^2_\alpha)\Big\}\\
&= \frac{1}{2}\sigma^2(\bar{t}_\alpha)\Big\{\alpha(\overline{x}_\alpha - \overline{y}_\alpha)^2 + \epsilon e^{\lambda(T-\hat{t}_\alpha)}(\overline{x}^2_\alpha + \overline{y}^2_\alpha) + 2\gamma(\overline{x}^2_\alpha - \overline{y}^2_\alpha)\Big\}.
\end{aligned}$$

Combining this with some straightforward calculation and the observation $\lim_{\gamma\to 0} I_4 \leq -r\delta$ we get the estimate

$$\limsup_{\gamma\downarrow 0,\alpha\uparrow\infty}(I_1+I_2+I_3+I_4) \leq \epsilon\Big(r + \frac{1}{2}\sigma^2(t_\epsilon) - d - \frac{\lambda}{2}\Big)e^{\lambda(T-t_\epsilon)}x_\epsilon^2 \leq 0 \tag{8.20}$$

by choosing λ large enough. It remains to estimate the non-local part $I_5 + I_6$. First note that $\tau^{\hat{x}_\alpha-\bar{x}_\alpha}_{\hat{t}_\alpha-\bar{t}_\alpha}\underline{v}^\gamma(\bar{t}_\alpha,\bar{x}_\alpha) = \underline{v}^\gamma(\hat{t}_\alpha,\hat{x}_\alpha)$ and $\tau^{\hat{y}_\alpha-\bar{y}_\alpha}_{\hat{t}_\alpha-\bar{t}_\alpha}\overline{v}^\mu_\gamma(\bar{t}_\alpha,\bar{y}_\alpha) = \overline{v}^\mu_\gamma(\hat{t}_\alpha,\hat{y}_\alpha)$. For I_5, the integrand equals

$$\begin{aligned}
&f_5 = \tau^{\hat{x}_\alpha-\bar{x}_\alpha}_{\hat{t}_\alpha-\bar{t}_\alpha}\underline{v}^\gamma(\bar{t}_\alpha,\bar{x}_\alpha e^z) - \tau^{\hat{y}_\alpha-\bar{y}_\alpha}_{\hat{t}_\alpha-\bar{t}_\alpha}\overline{v}^\mu_\gamma(\bar{t}_\alpha,\bar{y}_\alpha e^z) - (\underline{v}^\gamma(\bar{t}_\alpha,\bar{x}_\alpha) - \overline{v}^\mu_\gamma(\bar{t}_\alpha,\bar{y}_\alpha))\\
&-[\alpha(\hat{x}_\alpha-\hat{y}_\alpha)(\bar{x}_\alpha-\bar{y}_\alpha) + \epsilon e^{\lambda(T-\hat{t}_\alpha)}(\bar{x}_\alpha\hat{x}_\alpha + \bar{y}_\alpha\hat{y}_\alpha)](e^z-1)\\
&+[2\gamma(\hat{x}_\alpha - x_\alpha) - p + 2\gamma(\hat{y}_\alpha - y_\alpha) - \hat{p}](e^z-1)
\end{aligned}$$

To get a nicer expression than the above, we first take the limit $\gamma\downarrow 0$, make some estimates for the limit and then apply Fatou's lemma. Taking the limit and some computation reveals that

$$\begin{aligned}\limsup_{\gamma\downarrow 0} f_5 \leq & \underline{v}(t_\alpha, x_\alpha e^z) - \overline{v}^\mu(t_\alpha, y_\alpha e^z) - (\underline{v}(t_\alpha, x_\alpha) - \overline{v}(t_\alpha, y_\alpha)) \\ & - [\alpha(x_\alpha - y_\alpha)^2 + \epsilon e^{\lambda(T-t_\alpha)}(x_\alpha^2 + y_\alpha^2)^2](e^z - 1) \\ = & \Phi(t_\alpha, x_\alpha e^z, y_\alpha e^z) - \Phi(t_\alpha, x_\alpha, y_\alpha) + \psi(t_\alpha, x_\alpha, y_\alpha)(e^z - 1)^2 \\ \leq & \psi(t_\alpha, x_\alpha, y_\alpha)(e^z - 1)^2,\end{aligned}$$

where the last inequality follows by the maximality of $(t_\alpha, x_\alpha, y_\alpha)$. By Fatou's lemma we then have that

$$\limsup_{\gamma\downarrow 0} I_5 \leq \psi(t_\alpha, x_\alpha, y_\alpha) \int_{\mathbb{R}} (e^z - 1)^2 \nu_{t_\alpha}(dz)$$

For the non-standard term I_6, we begin by choosing γ small and α large so that $|\bar{x}_\alpha - \bar{y}_\alpha| \leq \frac{\delta}{2}$. Note also that by continuity and (8.16), we get

$$\underline{v}^\gamma(\hat{t}_\alpha, \hat{x}_\alpha) \geq \overline{v}^\mu_\gamma(\hat{t}_\alpha, \hat{y}_\alpha) + \delta > 0$$

for small γ, large α. Thus

$$\begin{aligned}g(\bar{y}_\alpha) - \overline{v}^\mu_\gamma(\hat{t}_\alpha, \hat{y}_\alpha) &= g(\bar{x}_\alpha) - \overline{v}^\mu(\hat{t}_\alpha, \hat{y}_\alpha) + (g(\bar{y}_\alpha) - g(\bar{x}_\alpha)) \\ &\geq g(\bar{x}_\alpha) - \overline{v}^\mu_\gamma(\hat{t}_\alpha, \hat{y}_\alpha) - \frac{\delta}{2} \\ &\geq g(\bar{x}_\alpha) - \underline{v}^\gamma(\hat{t}_\alpha, \hat{x}_\alpha) + \frac{\delta}{2}.\end{aligned}$$

By comparing possible values of H^* and H_* one then derives the estimates

$$\begin{aligned}I_6 \leq & \max\{0, H^*(g(\bar{x}_\alpha) - \underline{v}^\gamma(\hat{t}_\alpha, \hat{x}_\alpha))\big[(rK - d\bar{x}_\alpha - \overline{D}(\bar{t}_\alpha, \bar{x}_\alpha, \tau^{\hat{x}_\alpha - \bar{x}_\alpha}_{\hat{t}_\alpha - \bar{t}_\alpha} \underline{v}^\gamma))^+ \\ & - (rK - d\bar{y}_\alpha - \underline{D}(\bar{t}_\alpha, \bar{y}_\alpha, \tau^{\hat{y}_\alpha - \bar{y}_\alpha}_{\hat{t}_\alpha - \bar{t}_\alpha} \overline{v}^\mu_\gamma))^+\big]\} \\ \leq & d(\bar{y}_\alpha - \bar{x}_\alpha)^+ + \max\{0, H^*(g(\bar{x}_\alpha) - \underline{v}^\gamma(\hat{t}_\alpha, \hat{x}_\alpha))[\underline{D}(\bar{t}_\alpha, \bar{y}_\alpha, \tau^{\hat{y}_\alpha - \bar{y}_\alpha}_{\hat{t}_\alpha - \bar{t}_\alpha} \overline{v}^\mu_\gamma) \\ & - \overline{D}(\bar{t}_\alpha, \bar{x}_\alpha, \tau^{\hat{x}_\alpha - \bar{x}_\alpha}_{\hat{t}_\alpha - \bar{t}_\alpha} \underline{v}^\gamma)]\}.\end{aligned}$$

The first term vanishes when we take $\gamma \to 0, \alpha \to \infty$. By comparing possible values of the integrands we see that the second term is less than or equal to

$$\int_{|z|>\hat{\kappa}} \max\{0, -(\tau^{\hat{x}_\alpha - \bar{x}_\alpha}_{\hat{t}_\alpha - \bar{t}_\alpha} \underline{v}^\gamma(\bar{t}_\alpha, \bar{x}_\alpha e^z) - \tau^{\hat{x}_\alpha - \bar{y}_\alpha}_{\hat{t}_\alpha - \bar{t}_\alpha} \overline{v}^\mu_\gamma(\bar{t}_\alpha, \bar{y}_\alpha e^z)) + (\bar{x}_\alpha - \bar{y}_\alpha)e^z\}\nu_{\bar{t}_\alpha}(dz)$$

for some fixed $\hat{\kappa} > 0$. Defining

$$f_6 := \max\{0, 1_{|z|>\hat{\kappa}}[-(\tau^{\hat{x}_\alpha - \bar{x}_\alpha}_{\hat{t}_\alpha - \bar{t}_\alpha} \underline{v}^\gamma(\bar{t}_\alpha, \bar{x}_\alpha e^z) - \tau^{\hat{x}_\alpha - \bar{y}_\alpha}_{\hat{t}_\alpha - \bar{t}_\alpha} \overline{v}^\mu_\gamma(\bar{t}_\alpha, \bar{y}_\alpha e^z)) + (\bar{x}_\alpha - \bar{y}_\alpha)e^z]\}$$

and recalling calculations for f_5, we may estimate

$$\begin{aligned}\limsup_{\gamma\downarrow 0}(f_5 + f_6) \leq & \Phi(t_\alpha, x_\alpha e^z, y_\alpha e^z) - \Phi(t_\alpha, x_\alpha, y_\alpha) + \psi(t_\alpha, x_\alpha, y_\alpha)(e^z - 1)^2 \\ & + \max\{0, -1_{|z|>\hat{\kappa}}[\Phi(t_\alpha, x_\alpha e^z, y_\alpha e^z) + (x_\alpha - y_\alpha)e^z]\} \\ \leq & \max\{0, \psi(t_\alpha, x_\alpha, y_\alpha)(e^z - 1)^2 + 1_{|z|>\hat{\kappa}}(x_\alpha - y_\alpha)e^z\}\end{aligned}$$

where we have once again used maximality and nonnegativity of $\Phi(t_\alpha, x_\alpha, y_\alpha)$. Thus we have for I_5 and I_6 that

$$\begin{aligned}\limsup_{\gamma\downarrow 0}(I_5 + I_6) \leq & d(y_\alpha - x_\alpha)^+ + \psi(t_\alpha, x_\alpha, y_\alpha) \int_{\mathbb{R}} (e^z - 1)^2 \nu_{t_\alpha}(dz) \\ & + \max\{0, (x_\alpha - y_\alpha) \int_{|z|>\hat{\kappa}} e^z \nu_{t_\alpha}(dz)\}\end{aligned}$$

Thus

$$\limsup(I_5 + I_6) \leq \frac{\epsilon}{2} e^{\lambda(T-t_\epsilon)} \Big(\int_{\mathbb{R}} (e^z - 1)^2 \nu_{t_\epsilon}(dz) \Big) x_\epsilon^2 \tag{8.21}$$

as we take $\gamma \to 0$ and $\alpha \to \infty$, in this order. Combining (8.20) with (8.21) and taking $\gamma \to 0$ and $\alpha \to \infty$, we finally obtain the contradiction

$$0 < \mu \leq 0.$$

□

Since the viscosity solution v is both a sub- and supersolution, the comparison principle implies uniqueness for the semilinear Black and Scholes equation (1.1). We furthermore get that v is the smallest supersolution satisfying $v \geq g$, which is in lines with the classical characterization of value functions of optimal stopping problems. It is interesting to note that we do not need to assume $v \geq g$ a priori, but that this information is embedded in the operator and follows from the comparison principle.

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